Kalman Filters for Programmers

Alexander Wright

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1 Introduction

So, essentially, a Kalman filter predicts a future state given measurements of the current state which may be imprecise in nature due to noise or other errors. The positions are represented as gaussian probabilities that should sum to a total probability of one across the state space. The computations themselves are represented using matrices/linear algebra and correspond to the mathematics required to combine multiple gaussians. Our examples will be tailored to tracking a particle that moves with velocity and acceleration. The following equations may seem brief, but they are essentially the heart of the Kalman Filter.

2 Matrix Equations

Prediction Step

$x' = F \cdot x + u$

$P' = F \cdot P \cdot F^T$

Measurement Update

$y = z - H \cdot x$

$S = H \cdot P \cdot H^T + R$

$K = P \cdot H^T \cdot S^{-1}$

$x' = x + (K \cdot y)$

$P' = (I - K \cdot H) \cdot P$

Where the variables are:

x' – estimate

P' – uncertainty covariance

F – state transition matrix

u – external motion vector

z – measurement vector

H – measurement function

K – Kalman gain

S – project system uncertainty into measurement space

R – noise

I – Identity matrix

Since we have two stages, a prediction step and a measurement update, we have some constants and state variables that need to be maintained between iterations of the updates. The constants can be defined at initialization and will not require updates each time step.

Constants
H - measurement function
I - identity matrix
R - measurement uncertainty
F - state transition
u - external motion

State variables
Z - measurement update
y - error update
S - project system uncertainty into measurement space R is noise
K - Kalman gain
x - update estimate
P - update uncertainty (covariance)

The dimensions of the Kalman filter depend on the number of variables you wish to track. Each additional variable added increases the size of the matrix exponentially. The most important matrix in the set would be the "F" State Transition matrix, the "H" measurement function must be paired with the "F" matrix in such a way that the measurement aligns with the corresponding locations of the desired products.

3 Kinematic Equation of Motion under constant acceleration

Some basic background for particle motion under uniform acceleration:

$$x = x_0 + vt + \frac{1}{2}at^2 \tag{1}$$

So, we wish to represent this in the Kalman filter and we can do this in multiple ways. For a single dimension, we can use a 4x4 matrix for F and extend to multiple dimensions by running the same filter on each dimension independently. Once on X, and once on Y, and once on Z, etc. Since motion in X, Y, and Z are linearly independent of the others, things can be simplified by simply running a smaller filter multiple times. However, this requires more state tracking variables as you now have a set for each dimension.

3.1 Kalman Filter State Transition Matrix F in one dimension:

$$F = \begin{bmatrix} 1 & dt & dt2 \\ 0 & 1 & dt \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Note that dt represent a delta time between updates, and dt2 is short hand for $\frac{1}{2}dt^2$.

This assumes in Z is of the form:

$$Z = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \end{bmatrix} \tag{3}$$

However, we could also represent a two dimensional motion using a 6x6 matrix and only run the filter once, simultaneously solving for both X and Y.

3.2 Kalman Filter State Transition Matrix F in two dimensions:

$$F = \begin{bmatrix} 1 & dt & dt2 & 0 & 0 & 0 \\ 0 & 1 & dt & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & dt & dt2 \\ 0 & 0 & 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Note that this assumes the input vector Z is of the form:

$$Z = \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} \tag{5}$$

3.3 We can also extend this into three dimensions as follows:

$$F = \begin{bmatrix} 1 & dt & dt2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & dt & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & dt & dt2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & dt & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & dt & dt2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & dt \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Note that this assumes input vector Z is of the form:

$$Z = \begin{bmatrix} x \\ \dot{x} \\ \ddot{x} \\ y \\ \dot{y} \\ \ddot{y} \\ z \\ \dot{z} \\ \ddot{z} \end{bmatrix} \tag{7}$$

3.4 An alternate form of the 3D State Transition Matrix is as follows:

Kalman3D

$$F = \begin{bmatrix} 1 & 0 & 0 & dt & 0 & 0 & dt2 & 0 & 0 \\ 0 & 1 & 0 & 0 & dt & 0 & 0 & dt2 & 0 \\ 0 & 0 & 1 & 0 & 0 & dt & 0 & 0 & dt2 \\ 0 & 0 & 0 & 1 & 0 & 0 & dt & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & dt & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & dt \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Note that this assumes in input vector Z is of the form:

$$Z = \begin{bmatrix} x \\ y \\ z \\ \dot{x} \\ \dot{y} \\ \dot{z} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} \tag{9}$$

4 The remaining matrices for one dimension

$$H = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{10}$$

This essentially says we are measuring position, but not velocity or acceleration

$$P = \begin{bmatrix} 1000 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1000 \end{bmatrix} \tag{11}$$

This matrix essentially says we are uncertain (high value) of where we currently are, the equations will quickly converge to the correct value, but you may alternatively put in the initial positions into the H matrix, and set this to one along the diagnol for the values you entered (position, velocity, and/or acceleration)

$$u = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

This essentially says no external motion (such as gravity)

$$R = \begin{bmatrix} 1 \end{bmatrix} \tag{13}$$

Assuming no noise

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

Identity matrix for completeness

5 The remaining matrices for two dimensions

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{15}$$

This essentially says we are measuring position, but not velocity or acceleration

$$P = \begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix} \tag{16}$$

This matrix essentially says we are uncertain (high value) of where we currently are, the equations will quickly converge to the correct value, but you may alternatively put in the initial positions into the H matrix, and set this to one along the diagnol for the values you entered (position, velocity, and/or acceleration)

$$u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

This essentially says no external motion (such as gravity)

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

Assuming no noise

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{19}$$

Identity matrix for completeness

6 The remaining matrices for three dimensions

Remaining Kalman matrices in three dimensions:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

This essentially says we are measuring position, but not velocity or acceleration

$$P = \begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix} \tag{21}$$

This matrix essentially says we are uncertain (high value) of where we currently are, the equations will quickly converge to the correct value, but you may alternatively put in the initial positions into the H matrix, and set this to one along the diagnol for the values you entered (position, velocity, and/or acceleration)

$$u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{22}$$

This essentially says no external motion (such as gravity)

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{23}$$

Assuming no noise

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

Identity matrix for completeness

7 The corresponding matrices for three dimensions, with second vector ordering

Remaining Kalman matrices in three dimensions:

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{25}$$

This essentially says we are measuring position, but not velocity or acceleration

$$P = \begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 \end{bmatrix} \tag{26}$$

This matrix essentially says we are uncertain (high value) of where we currently are, the equations will quickly converge to the correct value, but you may alternatively put in the initial positions into the H matrix, and set this to one along the diagnol for the values you entered (position, velocity, and/or acceleration)

$$u = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{27}$$

This essentially says no external motion (such as gravity)

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{28}$$

Assuming no noise

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

Identity matrix for completeness